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In this report we present a block diagonalization theorem which is designed to study the stability and bifurcation of rotating systems, or more generally, of relative equilibria. The context of the discussion is the energy-momentum method for mechanical systems with symmetry. Simo, Posbergh and Marsden [1989] and Lewis and Simo [1989] discovered crucial special cases of the block diagonalization theorem for uniformly rotating systems, including general nonlinear elasticity and geometrically exact rods. ~~Our~~ The purpose here is to abstract these examples and prove a general geometric theorem. ~~We expect~~ These general results will be important for rotating gravitational fluid masses as well. (jhd)

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§1 Introduction

In this report we present a block diagonalization theorem which is designed to study the stability and bifurcation of rotating systems, or more generally, of relative equilibria. The context of the discussion is the energy-momentum method for mechanical systems with symmetry. Simo, Posbergh and Marsden [1989] and Lewis and Simo [1989] discovered crucial special cases of the block diagonalization theorem for uniformly rotating systems, including general nonlinear elasticity and geometrically exact rods. Our purpose is to abstract these examples and prove a general geometric theorem. We expect these general results will be important for rotating gravitational fluid masses as well.

For rotating systems the result says that a splitting of coordinates can be *explicitly* found on a linearized level which represent the rotational and internal vibrational modes. In these coordinates, the second variation of an augmented Hamiltonian is block diagonal. Of course coordinates can always be found in principle to do this, but we are able to do it explicitly enough to give useful stability and, we also believe, bifurcation criteria.

On the other hand, the symplectic form does *not* block diagonalize, indicating that the rotational and internal modes are in fact dynamically coupled. However, for purposes of the stability calculation, block diagonalization of the augmented energy is what is important. The off diagonal terms in the symplectic form (sometimes called *Coriolis coupling terms*) are, however, sufficiently simple that they should be useful for studying the dynamic interaction of the rotational and internal vibrational modes.

For rotating pseudo-rigid bodies, Lewis and Simo [1989] noticed that the computation of the definiteness of the second variation is considerably simplified by our result - in this case the simplification saves considerable computation time. In fact the symbolic and numerical manipulation required one to test a full 14×14 matrix for definiteness; block diagonalization reduces this to testing a 6×6 matrix for nonisotropic bodies and to a 3×3 matrix for the isotropic case.

§2 The Energy-Momentum Method

We begin our work in the context of standard mechanical systems with symmetry before any reductions have taken place. In other words, we begin with a *symplectic* manifold (P, Ω)

rather than a *Poisson* manifold. In fact, shortly we shall specialize to the case of $P = T^*Q$ and a Hamiltonian of the form kinetic plus potential.

Let G be a Lie group acting symplectically on P with an equivariant momentum mapping

$$\mathbf{J} : P \to \mathfrak{g}^* \quad . \tag{1}$$

(see Abraham and Marsden [1978], Marsden [1981] or Marsden *et al.* [1982] for the standard definitions and results used here).

Let $H : P \to \mathbb{R}$ be a given G-invariant Hamiltonian. A point z_e in P is called a ***relative equilibrium*** if there is a $\xi \in \mathfrak{g}$, the Lie algebra of G, such that for all $t \in \mathbb{R}$,

$$z(t) = \exp(t\xi)\, z_e, \tag{2}$$

where $z(t)$ is the dynamical orbit of X_H, the Hamiltonian vector field of H, with $z(0) = z_e$.

The energy-momentum method rests on the following result.

2.1 Relative Equilibrium Theorem *A point z_e is a relative equilibrium iff there is a $\xi \in \mathfrak{g}$ such that z_e is a critical point of $H_\xi : P \to \mathbb{R}$, where*

$$H_\xi(z) = H(z) - \langle \mathbf{J}(z) - \mu_e, \xi \rangle \tag{3}$$

and $\mu_e = \mathbf{J}(z_e)$.

In (3), the Lie algebra element $\xi \in \mathfrak{g}$ may be regarded as a Lagrange multiplier. Since $\mathbf{J}$ is conserved by the flow of X_H, the set $\mathbf{J} - \mu_e = 0$ is preserved, so one may regard it as a (non-holonomic) constraint set. It also follows that $\xi \in \mathfrak{g}_{\mu_e}$, the isotropy algebra of μ_e (with respect to the coadjoint action). Thus,

$$\delta H_\xi(z_e) = 0 \quad . \tag{4}$$

may be regarded as a (constrained)*variational principle for relative equilibria.*

The relative equilibrium theorem is readily verified. Of course it has a long history, going back to Lagrange and Poincaré for rotating systems. Like many basic results, it has been rediscovered in a number of contexts by various authors. Early references in our context are Arnold [1966], Smale [1970] and Marsden and Weinstein [1974]. As we shall state below, the

relative equilibrium theorem sometimes specializes to the ***principle of symmetric criticality*** (Palais [1979]).

The energy-momentum method proceeds as follows (see Holm *et al.* [1985] for the meaning of formal stability and related references).

Energy-Momentum Method

To test a relative equilibrium $z_e \in P$ for formal stability:

1 Choose $\xi \in \mathfrak{g}$ such that $\delta H_\xi(z_e) = 0$

2 Choose a linear subspace $S \subset T_{z_e}P$ such that

i $S \subset \ker dJ(z_e)$ and

ii S complements $T_{z_e}(G_{\mu_e} \cdot z_e)$ in $\ker TJ(z_e)$, where $G_{\mu_e} \subset G$ is the isotropy subgroup of μ_e.

3 Test

$$\delta^2 H_\xi(z_e)$$

for definiteness as a bilinear form on S.

The energy-momentum method "covers" the energy-Casimir method (Holm *et al.* [1985]) in the sense that if the latter applies and gives formal stability, so does the former. One difficulty with the energy-Casimir method is that on the reduced space P/G, there may not be enough Casimirs to make the method effective, even to get the analogue $\delta(H + C) = 0$ of (4). This difficulty is genuine for the case of geometrically exact rods, for instance. See Simo, Posbergh and Marsden [1989] for further details.

The fact that $\delta^2 H_\xi(z_e)$ drops to the reduced space follows from the next lemma.

2.2 Gauge Invariance Lemma

$$\delta^2 H_\xi(z_e)(\eta_P(z_e), \delta z) = 0 \tag{5}$$

for all $\delta z \in \ker TJ(z_e)$ *and* $\eta \in \mathfrak{g}$, *where* η_P *denotes the infinitesimal generator of the group action on* P.

This follows readily from invariance of H and equivariance of $\mathbf{J}$. One can view (5) as a block diagonalization result on the unconstrained tangent space $T_{z_e}P$, but it does not yield block diagonalization *within* the constrained subspace $\mathcal{S}$ in the energy-momentum method. It is the latter that we are concerned with.

One can identify any choice of $\mathcal{S}$ with the tangent space to the ***reduced space***

$$P_{\mu_e} = J^{-1}(\mu_e)/G_{\mu_e}$$

at $[z_e]$ (assuming, as we shall, that μ_e is a regular and generic value; c.f. Weinstein [1984]). However, it is easier to do our analysis directly on $T_{z_e}T^*Q$ rather than on the quotient space. This is the usual situation found in constrained optimization problems. However, dropping the calculations to the quotient space at the appropriate point will play a useful role.

§3 Simple Mechanical Systems

We now specialize to the systems we will be studying. Let Q be a configuration manifold and $P = T^*Q$ with its canonical symplectic structure and cotangent coordinates (q^i, p_i) in the finite dimensional case. {Whenever we use coordinates, we assume Q is finite dimensional, although the results are *not* restricted to this case.} Coordinates on the velocity phase space TQ are similarly denoted $(q^i, \dot{q}^i)$.

Let g denote a Riemannian metric on Q; in coordinates we write the components of g as g_{ij} as usual, and we write g^{ij} for the inverse tensor. Let $K : TQ \to \mathbb{R}$ denote the corresponding kinetic energy, i.e.,

$$K(q, \dot{q}) = \frac{1}{2} g_{ij}(q)\, \dot{q}^i\, \dot{q}^j \,. \tag{1}$$

Let $V : Q \to \mathbb{R}$ be a given potential.

Assume G acts on Q (by a left action) and hence on T^*Q by the cotangent lift, so the equivariant momentum map is given by

$$\langle \mathbf{J}, \xi \rangle(\alpha_q) = \langle \alpha_q, \xi_Q(q) \rangle. \tag{2}$$

In coordinates, we define the ***action coefficients*** $A_a^i(q)$ by writing

$$[\xi_Q(q)]^i = A^i_a(q)\,\xi^a \tag{3}$$

where a, b, c, ... denote coordinate indices for the Lie algebra $\mathfrak{g}$. Thus (2) becomes

$$J_a(q, p) = p_i\, A^i_a(q). \tag{4}$$

We assume that G acts on Q by isometries and that the potential V is G-invariant. For elasticity, for instance, this is the requirement of material frame indifference. Note that (3) of §2 reads

$$H_\xi(q, p) = \tfrac{1}{2} g^{ij} p_i p_j + V(q) - p_i A^i_a(q)\,\xi^a. \tag{5}$$

Define the ***moment of inertia tensor*** $\mathbb{I}$ for the system ***locked at*** $q \in Q$ by

$$\mathbb{I}_{ab}(q) = g_{ij}(q)\, A^i_a(q)\, A^j_b(q) \tag{6}$$

(alternatively, in terms of the q-dependent inner product $\langle \xi, \eta \rangle := \langle \xi_Q(q), \eta_Q(q) \rangle$ on $\mathfrak{g}$, we have $\langle \xi, \eta \rangle = \mathbb{I}_{ab}(q)\xi^a\eta^b$), and define the ***augmented potential*** V_ξ by

$$V_\xi(q) = V(q) - \tfrac{1}{2}\mathbb{I}_{ab}(q)\,\xi^a\xi^b. \tag{7}$$

One can readily verify the following (see Abraham and Marsden [1978] and Palais [1979]) by writing out the conditions $\delta H_\xi = 0$ in **2.1**. A more elegant argument is, however, given below.

3.1 Principle of Symmetric Criticality *A point* $z_e = (q^i, p_j)$ *is a relative equilibrium if and only if there is a* $\xi \in \mathfrak{g}_{\mu_e}$ *such that*

i $p_i = g_{ij} A^j_a \xi^a$ (*i.e.*, p_e *is the Legendre transform of* $\xi_Q(q_e)$) (8a)

and

ii q^i *is a critical point of* V_ξ.(*i.e.*, $\dfrac{\delta}{\delta q} V_\xi \Big|_{q = q_e} = 0.$) (8b)

This is useful for carrying out the computations that follow. We also observe that V_ξ is G_ξ-invariant, and so induces a function on Q/G_ξ.

Define the one-form A^ξ on Q by

$$A_i^{\xi}(q) = g_{ij}(q)\, A_a^j(q)\, \xi^a \tag{9}$$

or abstractly, $A^{\xi}(q) = [\xi_Q(q)]^{\flat}$, where $^{\flat}$ denotes the index lowering operation with respect to the metric g_{ij}. In other words, $A^{\xi}(q)$ is the Legendre transform of $\xi_Q(q)$. We remark that A may be viewed as a G-connection for the bundle $Q \to Q/G$ and that this connection plays an important role in Berry's phase; cf. Marsden, Montgomery and Ratiu [1988]. Now notice that at equilibrium, (8a) says

$$p_e = A^{\xi}(q_e)\,. \tag{10}$$

Also note that

$$H_{\xi}(q, p) = K_{\xi}(q, p) + V_{\xi}(q, p) + \langle \mu_e, \xi \rangle \tag{11}$$

where $K_{\xi}(q, p) = \frac{1}{2} \| p - A^{\xi}(q) \|^2$, and V_{ξ} is given by (7). By (10), K_{ξ} has a critical point at z_e. Thus, (8b) is a direct consequence of the relative equilibrium theorem and (11).

In the energy-momentum method we shall use a special choice of S, namely

$$S = \left\{ v_{z_e} \in T_{z_e}T^*Q \mid T\pi_Q \cdot v_{z_e} \text{ is g-orthogonal to } T(G_{\mu_e} \cdot q_e) \text{ and } v_{z_e} \in \ker[TJ(z_e)] \right\}. \tag{12}$$

Letting coordinates on TT^*Q be denoted

$$(q^i, p_i, \delta q^i, \delta p_i),$$

(12) reads, with the help of (8a),

$$S = \Big\{ (q^i, p_i, \delta q^i, \delta p_i) \mid g_{ij}(\delta q)^i A_a^j \chi^a = 0 \text{ for all } \chi \in \mathfrak{g}_{\mu_e} \text{ and}$$
$$(\delta p)_i A_a^i + g_{ij} A_b^j \xi^b \frac{\partial A_a^i}{\partial q^k} (\delta q)^k = 0 \Big\} \tag{12'}$$

§4 Rigid Variations

One version of the cotangent bundle reduction theorem (see Abraham and Marsden [1978] and Kummer [1981], Montgomery [1986] and references therein) states that the reduced space

$(T^*Q)_{\mu_e}$ is a symplectic bundle over $T^*(Q/G)$ with fiber the coadjoint orbit through μ_e. Thus there is an isomorphism

$$T_{[z_e]}(T^*Q)_{\mu_e} \cong \mathfrak{g}/\mathfrak{g}_e \times T_{[z_e]}(T^*(Q/G)) \cong \mathfrak{g}/\mathfrak{g}_{\mu_e} \times (U_{INT} \times U_{INT}^*)$$

where U_{INT} is a model space for Q/G. For $G = \mathbf{SO}(3)$, U_{INT} *models the configuration space for the internal modes*, while $\mathfrak{g}/\mathfrak{g}_e \cong T_{\mu_e}O_{\mu_e}$ *models the phase space for rigid modes*. Our goal is to realize this decomposition explicitly, in such a way that $\delta^2 H_\xi(z_e)$ block diagonalizes. The bundle $(T^*Q)_\mu \to T^*(Q/G)$ with fiber O_μ also has a natural connection (Montgomery [1986]) and our decomposition should be related in some way to the horizontal-vertical split for this connection. *However we proceed directly here*; see also the comments in §5 below.

We will define two subspaces S_{RIG} and S_{INT} of S and further subspaces U_{INT} and U_{INT}^ξ of S_{INT} such that

$$S = S_{RIG} \oplus S_{INT} \cong S_{RIG} \oplus (U_{INT} \oplus U_{INT}^\xi) \tag{1}$$

relative to which $\delta^2 H_\xi(z_e)$ will be block diagonal. As above, the first component $S_{RIG} \cong \mathfrak{g}/\mathfrak{g}_{\mu_e}$ of S is isomorphic to the tangent space to the coadjoint orbit through μ_e. As we shall see, this component will also carry the coadjoint orbit symplectic structure. This first component is defined in terms of rigid variations as follows: Let

$$\mathfrak{g}_Q = \{\eta_Q(q) \in TQ \mid \eta \in \mathfrak{g} \text{ and } q \in Q\} \tag{2}$$

and let $T\mathfrak{g}_Q \subset TTQ$ be its tangent bundle.

4.1 Definition *Let* $V_{RIG} = s(T\mathfrak{g}_Q)$ *where* $s : T^2Q \to T^2Q$ *is the canonical involution. Alternatively,* V_{RIG} *consists of double tangents of curves denoted by* $\Delta\dot{q}$ *(identified with velocity variations of superposed rigid body motions in the case of* $\mathbf{SO}(3)$*)*

$$\Delta\dot{q} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{d}{dt}\Big|_{t=0} \exp(\varepsilon\eta(t))\, q(t)$$

where $\eta(t)$ *is a curve in* $\mathfrak{g}$ *with* $\eta(0) = \eta$ *and* $q(t)$ *is a curve in* Q. (The canonical involution in effect swaps the order of differentiation.)

In coordinates, if we write elements of V_{RIG} as

$$(q^i, \dot{q}^i, \Delta q^i, \Delta \dot{q}^i), \tag{3a}$$

then we find that

$$\Delta q^i = A^i_a \eta^a \quad \text{and} \quad \Delta \dot{q}^i = \frac{\partial A^i_a}{\partial q^k} \dot{q}^k \eta^a + A^i_a \zeta^a. \tag{3b}$$

Now let $\mathbb{FL} : TQ \to T^*Q$ be the Legendre transform given by

$$p_i = g_{ij} \dot{q}^j \tag{4}$$

and let $T\mathbb{FL} : TTQ \to TT^*Q$ be its tangent map. Set

$$\mathcal{S}_{RIG} = T\mathbb{FL} \cdot V_{RIG} \cap \mathcal{S}, \tag{5}$$

where $\mathcal{S}$ is defined by (12) of §3. If we let $\mathfrak{g}^{\perp}_{\mu_e}$ denote the (q-dependent) orthogonal complement of $\mathfrak{g}_{\mu_e}$ in the metric $\mathbb{I}_{ab}$, then one finds that $\mathcal{S}_{RIG}$ is parametrized by elements $\eta \in \mathfrak{g}^{\perp}_{\mu_e}$ as follows: we write elements of $\mathcal{S}_{RIG}$ as

$$(q^i, p_j, \Delta q^i, \Delta p_j) \tag{6a}$$

where

$$\Delta q^i = A^i_a \eta^a \tag{6b}$$

and

$$\Delta p_i = -\frac{\partial A^k_a}{\partial q^i} \eta^a p_k + g_{ij} A^j_a \zeta^a \tag{6c}$$

where $\eta \in \mathfrak{g}^{\perp}_{\mu_e}$ and where $\zeta \in \mathfrak{g}$; the condition that (6a) belongs to $\ker(T_{z_e}\mathbf{J})$ is equivalent to the relation

$$\zeta^a = \mathbb{I}^{ab}(\mathrm{ad}^*_\eta \mu_e)_b \tag{7}$$

i.e., $\xi^b = \mathrm{ad}^*_\eta \mu_e$, so ζ is determined by η. One checks that $\eta \in \mathfrak{g}^{\perp}_{\mu_e}$ as well.

§5 The Internal Vibration Space

Now we define a complement to $\mathcal{S}_{RIG}$ in $\mathcal{S}$. We will do this by a constructive procedure that can be effectively carried out in examples. As we have mentioned, this complement appears to be not the same as, but related to the complement to the vertical space relative to a natural connection on the coadjoint orbit bundle $(T^*Q)_\mu \to T^*(Q/G)$. In this regard we note that the metric naturally induced on Q/G is Wilson's G-matrix (see Wilson, Decius and Cross [1955]). Our decomposition appears to be finer than the one proposed by Guichardet [1984] and discussed by Iwai [1988]. Notice that we have connections on all levels of this *tower of bundles*

$$T^*Q \supset J^{-1}(\mu) \to (T^*Q)_\mu \to T^*(Q/G)$$

where $J^{-1}(\mu) \to (T^*Q)_\mu$ is regarded as a G_μ bundle and $(T^*Q)_\mu \to T^*(Q/G)$ is regarded as an O_μ bundle, where O_μ is the coadjoint orbit through μ.

The Guichardet-Iwai results appear to be largely concerned with the bundle $J^{-1}(\mu) \to (T^*Q)_\mu$; the fact that the reduced space $(T^*Q)_\mu$ still has the factor O_μ seems to be the reason the connection on the G_μ bundle $J^{-1}(\mu) \to (T^*Q)_\mu$ is not sufficient to completely isolate the vibrational modes from the rotational ones. We believe that the O_μ bundle fills this gap. These remarks aside, we turn to the explicit construction of $\mathcal{S}_{INT}$. To do this, we first describe U_{INT}.

Recall that the ***augmented potential*** V_ξ is given by

$$V_\xi = V + L_\xi \tag{1a}$$

where

$$L_\xi(q) = -\tfrac{1}{2}\langle \xi_Q(q), \xi_Q(q)\rangle . \tag{1b}$$

For mechanical systems undergoing stationary rotations about ξ, i.e., $G = \mathbf{SO}(3)$ and G_{μ_e} = rotations about the axis μ_e which is parallel to ξ), we note that L_ξ gives the potential of the centrifugal force. Now define U_{INT} as the subspace on which V_ξ or equivalently L_ξ ***looks objective*** in the sense of nonlinear elasticity (cf. Marsden and Hughes [1984]). More precisely:

5.1 Definition

$$U_{INT} = \{\delta q \in T_{q_e}Q \mid \langle \delta q, (\mathbf{L}_{\eta_Q} \mathbf{d} L_\xi)(q)\rangle = 0 \text{ for all } \eta \in \mathfrak{g}_{\mu_e}^{\perp} \text{ and } \langle \delta q, \chi_Q(q_e)\rangle = 0 \text{ for all } \chi \in \mathfrak{g}_{\mu_e}\} \quad (2)$$

where the first pairing is the natural pairing between vectors and one forms while the second is the metric inner product.

Since V_ξ has a critical point at q_e (by the principal of symmetric criticality) and V is G-invariant, we find that

$$\langle \delta q, (\mathbf{L}_{\eta_Q} \mathbf{d} C_\xi)(q)\rangle = \delta^2 V_\xi(q_e)(\delta q, \eta_Q(q)) \quad (3)$$

and so we see that the geometric condition (2) is exactly what is needed to block diagonalize $\delta^2 V_\xi(q_e)$ within $\mathcal{S}$. In coordinates, the first condition on δq^i defining U_{INT} is the geometric condition

$$\delta q^i \eta^a \xi^b \xi^c \frac{\partial}{\partial q^i}\left[A_a^k \frac{\partial}{\partial q^k}(A_b^\ell A_c^m g_{\ell m})\right] = 0; \quad (2')$$

the second condition is just the defining condition on $\mathcal{S}$. Now we are ready to define $\mathcal{S}_{INT}$.

5.2 Definition $\mathcal{S}_{INT} = \{\delta z \in T_{z_e}T^*Q \mid \delta q \in U_{INT} \textit{ and } \delta z \in \ker[TJ(z_e)]\} \subset \mathcal{S}.$ (4)

5.3 Proposition $\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$.

This is easy to check. The idea is that $\mathcal{S}_{RIG} \cap \mathcal{S}_{INT} = \{0\}$, that $\dim \mathcal{S}_{RIG} = \dim(\mathfrak{g}/\mathfrak{g}_\mu)$ and that $\mathcal{S}_{INT}$ is determined by $\dim(\mathfrak{g}/\mathfrak{g}_\mu)$ equations. Also, we write

$$\mathcal{S}_{INT} \cong U_{INT} \oplus U_{INT}^{\xi} \quad (5)$$

where $U_{INT}^{\xi} = \{\delta p - A^\xi(q_e) \mid \delta p \in U_{INT}^*\}$ is the dual space with a momentum shift by A^ξ (see equation (10) of §4). The relation (5) is really a coordinate description; to do it intrinsically, we use the metric connection to split $T_{z_e}T^*Q = T_{q_e}Q \oplus (T_{q_e}Q)^*$ (this split is in fact nothing more than what we do in coordinates to identify accelerations and momenta with vectors) then we take the

horizontal and vertical splitting of U_{INT} in $\mathcal{S}_{INT}$, with the vertical component followed by the momentum shift by $A^{\xi}(q_e)$.

We remark here that even if G is abelian (for instance, $G = S^1$ in the case of planar coupled rigid bodies) then the decompositions are not trivial: while $\mathcal{S}_{RIG} = \{0\}$ in this case, $\mathcal{S}_{INT} = U_{INT} \oplus U_{INT}^{\xi}$ is still not a trivial decomposition.

Now $H_\xi = K_\xi + V_\xi + \langle \mu_e, \xi \rangle$ and we have arranged for V_ξ to be block diagonal. As far as K_ξ is concerned, we compute in coordinates that

$$K_\xi = \frac{1}{2} g^{ij}(p_i - g_{ik}A_a^k\xi^a)(p_j - g_{jm}A_b^m\xi^b). \tag{6}$$

Thus, since $p_i = g_{ik} A_a^k \xi^a$ at equilibrium, we get

$$\delta^2 K_\xi(z_e) \cdot (\delta z, \delta \bar{z}) = g^{ij}\, \delta p_i\, \delta \bar{p}_j\,. \tag{7}$$

It is clear that $\delta^2 K_\xi$ block diagonalizes from $\delta^2 V_\xi$ within $U_{INT} \oplus U_{INT}^{\xi}$ by construction. Regarding the block diagonalization of $\delta^2 K_\xi$ on $\mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$, we shall use some further interesting identities.

First, here is an equivalent characterization of U_{INT} in terms of superposed motions:

5.4 Proposition *Let $q_e \in Q$ be a curve tangent to δq at q_e, let $\eta \in \mathfrak{g}_{\mu_e}^{\perp}$ and let $\eta_\varepsilon = \mathrm{Ad}_{\exp(\varepsilon\xi)}\eta$. Then U_{INT} is characterized by those δq orthogonal to $T_{q_e}(G_{\mu_e} \cdot q_e)$ and satisfying*

$$\frac{d}{d\varepsilon}\langle \xi_Q(q_\varepsilon), (\eta_\varepsilon)_Q(q_\varepsilon)\rangle\Big|_{\varepsilon=0} = 0 \tag{8a}$$

or, equivalently,

$$\frac{d}{d\varepsilon}\langle \xi_Q(\exp(\varepsilon\xi)q_\varepsilon)\,,\ \eta_Q(\exp(\varepsilon\xi)q_\varepsilon)\rangle\Big|_{\varepsilon=0} = 0\,. \tag{8b}$$

This is verified by a direct coordinate calculation. We can lift this expression to get an alternative characterization of $\mathcal{S}_{INT}$. We consider the momentum map $\mathbf{J}$ restricted to $\mathfrak{g}_{\mu_e}^{\perp}$ and regarded as a function on TQ. In other words, for $\zeta \in \mathfrak{g}_{\mu_e}^{\perp}$, set

$$J(\zeta)(\delta q) = \langle \zeta_Q(q), \delta q \rangle = g_{ij} A_a^i \zeta^a (\delta q)^j. \tag{9}$$

Now consider the condition

$$\frac{d}{dt} J(\zeta)(\delta q)\Big|_{t=0} = 0 \tag{10}$$

where ζ is to evolve as $\dot{\zeta} = [\xi, \zeta]$ which is consistent with (8a) and $\zeta \in \mathfrak{g}_{\mu_e}^{\perp}$; here ξ is the Lie algebra element giving the relative equilibrium. Equation (10) defines a condition on T(TQ). We shall regard it as a condition on $T_{z_e}(T^*Q)$ via the Legendre transform. For simplicity we still write the resulting condition as $\dot{J} = 0$.

5.5 Proposition

$$\mathcal{S}_{INT} = \{\dot{J}(z_e) = 0\} \cap \mathcal{S}. \tag{11}$$

The condition $TJ(z_e) \cdot \delta z = 0$ reads

$$\delta p_i A_a^i(q) + p_i \frac{\partial A_a^i}{\partial a^k} dq^k = 0 \tag{12}$$

and using this, one can express the conditions defining $\mathcal{S}_{INT}$ entirely in terms of δq. This recovers the space U_{INT}, which models $T_{[q_e]}(Q/G)$, and then one gets, as before,

$$\mathcal{S}_{INT} = U_{INT} \oplus U_{INT}^{\xi}.$$

§ 6 Block Diagonalization

The block diagonalization results for $\delta^2 H_\xi$ follow from two basic formulas:

6.1 Proposition *Let* $\Delta z \in \mathcal{S}_{RIG}$ *and* $\delta z \in T_{z_e} P$. *Then*

$$\delta^2 H_\xi(z_e)(\Delta z, \delta z) = \frac{d}{dt} \langle \zeta_Q(q), \delta q \rangle - \langle [\xi, \eta], \delta J(z_e) \cdot \delta z \rangle \tag{1}$$

where Δz *has associated* η *and* ζ *as in* (3b) *and* (7) *of* §4.

6.2 Proposition *Let* δz_1 *and* $\delta z_2 \in S_{INT}$; *then*

$$\delta^2 H_\xi(z_e)(\delta z_1, \delta z_2) = \delta^2 K_\xi(z_e) \cdot (\delta z_1, \delta z_2) + \delta^2 V_\xi(q_e)(\delta q_1, \delta q_2) \tag{2}$$

Proposition **6.1**, which is proved by direct calculation, shows that $\delta^2 H_\xi(z_e)$ *block diagonalizes* on $S_{RIG} \oplus S_{INT}$, i.e., if $\Delta z \in S_{RIG}$ and $\delta z \in S_{INT}$, then

$$\delta^2 H_\xi(z_e)(\Delta z, \delta z) = 0 . \tag{3}$$

Proposition **6.2** then follows from our earlier calculations. It also follows that if $\Delta z \in S_{RIG}$ and $\Delta \bar{z} \in S_{RIG}$, then

$$\delta^2 H_\xi(z_e)(\Delta z, \Delta \bar{z}) = \frac{d}{dt}\langle \zeta_Q(q), \bar{\zeta}_Q \rangle \tag{4}$$

which is a generalization of the rigid body second variation formula for motion on the coadjoint orbit O_{μ_e} with the metric $\mathbb{I}_{ab}$. We summarize:

6.3 Theorem *The relative equilibrium* z_e *is formally stable* (*with* $\delta^2 H_\xi(z_e)$ *on* S *positive definite*) *if and only if*

i $\frac{d}{dt}\langle \zeta_Q(q), \bar{\zeta}_Q(q) \rangle$ *is positive definite on* S_{RIG}

and **ii** $\delta^2 V_\xi(q_e)$ *is positive definite on* U_{INT}.

We note that $\delta^2 V_\xi(q_e)$ separates (in coordinates on U_{INT}) into $\delta^2 V(q_e)$ plus a term quadratic in ξ. Thus, **ii** is implied by a condition of the form $\| \xi \| \leq \sqrt{\lambda_{min}}$, where $\| \ \|$ is a suitable norm and λ_{min} is the minimum (non-zero) eigenvalue of $\delta^2 V(q_e)$; one has to take care here that V itself does not have a critical point at q_e, so $\delta^2 V(q_e)$ does *not* make intrinsic sense. To see how this works in examples, see Simo, Posbergh and Marsden [1989] and Lewis and Simo [1989].

As far as the symplectic form Ω is concerned, we have

6.4 Theorem *Let* $\Delta z \in \mathcal{S}_{RIG}$ *and* $\delta z \in T_{z_e} P$. *Then*

$$\Omega(z_e)(\Delta z, \delta z) = -\langle \eta, \delta J(z_e) \cdot \delta z \rangle + \langle \zeta_Q(q_e), \delta q \rangle. \tag{5}$$

Notice that in an appropriate sense, $\delta^2 H_\xi(z_e)$ on $\mathcal{S}_{RIG} \times T_{z_e} P$ is the time derivative of the symplectic form Ω!

From (5) and (7) of §4 one finds that on $\mathcal{S}_{RIG} \times \mathcal{S}_{RIG}$, Ω gives the coadjoint orbit symplectic form

$$\Omega(z_e)(\Delta z, \Delta \bar{z}) = -\langle \mu_e, [\eta, \bar{\eta}] \rangle, \tag{6}$$

while on $\mathcal{S}_{RIG} \times \mathcal{S}_{INT}$ we have the cross terms

$$\Omega(z_e)(\Delta z, \delta z) = \langle \zeta_Q(q_e), \delta q \rangle, \tag{7}$$

which depend on the δq components alone.

We can summarize the situation with the following matrices:

$$\mathfrak{g} \,/\, \mathfrak{g}_{\mu_e} \qquad\qquad U_{INT}$$

U^{ξ}_{INT}

$$\delta^2 H_\xi(z_e) = \begin{bmatrix} \begin{bmatrix} \text{Generalized} \\ \text{Rigid Body} \\ \text{Second Variation} \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} d^2 V_\xi(z_e) & 0 \\ 0 & \delta^2 K_\xi(z_e) \end{bmatrix} & \\ 0 & & \end{bmatrix}$$

for the amended energy, and

$\mathfrak{g} / \mathfrak{g}_{\mu_e}$ U_{INT}

U^{ξ}_{INT}

$$\Omega = \begin{bmatrix} \left[\begin{array}{c}\text{Coadjoint Orbit}\\ \text{Symplectic Form}\end{array}\right] & \left[\begin{array}{c}\text{Internal-}\\ \text{Rigid Coupling (7)}\end{array}\right] & 0 \\ -\left[\begin{array}{c}\text{Internal-}\\ \text{Rigid Coupling (7)}\end{array}\right] & \left[\begin{array}{c}\text{Canonical symplectic form}\\ \text{plus a "magnetic" term}\end{array}\right] & \\ 0 & & \end{bmatrix}$$

For information on the "magnetic" term, and its interpretation as a curvature, we refer the reader to Kummer [1981]. Also, the coupling terms can be interpreted in terms of the curvature of the connection on the coadjoint bundle $T^*Q \to T^*(Q/G)$; see Montgomery [1986] and Lewis, Marsden, Montgomery and Ratiu [1986].

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